Complex Variables

Laplace Transform – Z Transform

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Outline

Outline

Complex plane (z-plane)

Complex plan is the plane equipped with the direct orthonormal basis $(O; u, v)$. La correspondence

$$\left\{ \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{C} \\ (x, y) & \mapsto & z = x + iy \end{array} \right.$$

is bijective.
By a slight abuse of notation, the point $M\,(x, y)$ and its affix $z = x + iy$ coincide.
If $z \neq 0$, the representation of the complex number z under the form modulus/argument is written

$$z = \rho e^{i\theta}$$

where $\rho = |z| = OM$ is the modulus z and $\theta = \arg z$ is a angle measure $\left(u, \overrightarrow{OM}\right)$ (in rad) defined modulo 2π, i.e., $\pm 2k\pi$, $k \in \mathbb{Z}$.

Complex function of the z-variable

For any function f of complex variable

$$f : \left\{ \begin{array}{rcl} \mathbb{C} & \rightarrow & \mathbb{C} \\ z \ = x + iy & \mapsto & f(z) = P(x,y) + iQ(x,y) \end{array} \right.$$

we can define a function F :

$$F : \left\{ \begin{array}{rcl} \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ (x,y) & \mapsto & F(x,y) = (P(x,y), Q(x,y)) \end{array} \right.$$

Outline

Limits - continuity

$\mathbb{C}$ is a vector space on $\mathbb{R}$ equipped with the norm $\|z\| = |z|$.
Let f define a complex variable function and $z_0 = x_0 + iy_0$ and l two complex numbers.

Definition: limit

$$\lim_{z \longrightarrow z_0} f(z) = l \text{ or } \; f(z) \underset{z \longrightarrow z_0}{\longrightarrow} l$$

means:

$$\forall \varepsilon > 0, \quad \exists \eta > 0, \quad |z - z_0| < \eta \Longrightarrow |f(z) - l| < \varepsilon$$

Definition: continuity

$$\begin{aligned} f \text{ continue at } z_0 \quad & \Longleftrightarrow \quad \lim_{z \longrightarrow z_0} f(z) = f(z_0) \\ & \Longleftrightarrow \quad P(x, y) \text{ and } Q(x, y) \text{ continue at } (x_0, y_0) \end{aligned}$$

Limits - continuity

Without any demonstration, we will admit that the standard operations on the limits or continuous functions are the same as those obtained for functions from $\mathbb{R}^2 \to \mathbb{R}$ or from $\mathbb{R} \to \mathbb{R}$

Warning !

If $P(x, y)$ is continuous at the point (x_0, y_0), then

$$\begin{cases} x \mapsto P(x, y_0) & \text{is continuous at} \quad x = x_0 \\ y \mapsto P(x_0, y) & \text{is continuous at} \quad y = y_0 \end{cases}$$

The reciprocal is wrong!

Complex infinity

The complex infinity denoted ∞ is the unique complex number ensuring the following properties with $a \in \mathbb{C}$:

$$\begin{aligned} \infty \times \infty &= \infty, |\infty| = \infty \\ \infty / a &= \infty, a/\infty = 0, a \times \infty = \infty \end{aligned}$$

- Representation on the Poincaré sphere,
- Extensions of the limit and neighboring definitions around infinity.

Outline

Algebraic functions

Functions	Définition	Continuiy	Associated T_{G}
$z \longmapsto z + a$	$\mathbb{C}$	$\mathbb{C}$	Translation
$z \longmapsto a\ z$	$\mathbb{C}$	$\mathbb{C}$	Similarity
$z \longmapsto \frac{1}{z}$	$\mathbb{C}^*$	$\mathbb{C}^*$	Inversion then symetry Ox
$z \longmapsto \frac{az+b}{cz+d}$	$\mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$	$\mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$	...

Outline

Exponential function

Definition

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Properties

$$\begin{aligned} e^z\big|_{z=x} &= e^x \\ e^{z_1+z_2} &= e^{z_1} e^{z_2} \\ e^{x+iy} &= e^x \left(\cos y + i \sin y\right) \\ e^{-z} &= \frac{1}{e^z} \end{aligned}$$

We have the same functional relations as in $\mathbb{R}$.

Hyperbolic and trigonometric functions

Hyperbolic functions

$$ch\ z = \frac{e^z + e^{-z}}{2}, \quad sh\ z = \frac{e^z - e^{-z}}{2}, \quad thz = \frac{shz}{chz}$$

Example: resolution of $ch\ z = 0$.

Trigonometric functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \tan z = \frac{\sin z}{\cos z}$$

Example: resolution of $sin\ z = 2$.

Hyperbolic and trigonometric functions

Properties

Functions	Definition set	Continuity set
exp	$\mathbb{C}$	$\mathbb{C}$
ch	$\mathbb{C}$	$\mathbb{C}$
sh	$\mathbb{C}$	$\mathbb{C}$
th	$\mathbb{C}\backslash\left\{i\left(\frac{\pi}{2}+k\pi\right), k\in\mathbb{Z}\right\}$	$\mathbb{C}\backslash\left\{i\left(\frac{\pi}{2}+k\pi\right), k\in\mathbb{Z}\right\}$
cos	$\mathbb{C}$	$\mathbb{C}$
sin	$\mathbb{C}$	$\mathbb{C}$
tan	$\mathbb{C}\backslash\left\{\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\right\}$	$\mathbb{C}\backslash\left\{\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\right\}$

Changing rules

$$\left\{\begin{array}{c}\cos iz = ch\ z\\ \sin iz = i\ sh\ z\\ \tan\ iz = i\ th\ z\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{c}ch\ iz = \cos z\\ sh\ iz = i\sin z\\ th\ iz = i\ \tan\ z\end{array}\right.$$

Outline

Multivalued function

To any z of $\mathbb{C}$, a unique value of e^z corresponds. However, to any z of $\mathbb{C}^*$, an infinity of values of $\arg z$ corresponds. To distinguish between these two cases, we are defining the so-called mono-valued vs. multivalues functions.

Definitions

- A function f is named **mono-valued** if to any value z a unique value of $f(z)$ corresponds.
- A function f is named **multi-valued** (aka multifunctions) if to any value z several distinct values of $f(z)$ correspond.

Multivalued functions

Examples

- The argument function :

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{R} \\ z & \longmapsto & \arg z \end{array}$$

  is a multi-valued function.
- The functions introduce earlier are mono-valued.

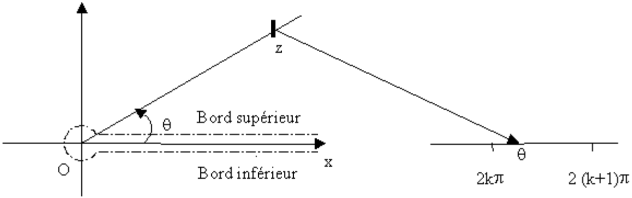
Study of the multivalued functions

To study the multivalued functions, we make them mono-valued by defining its restrictions (or "branches") of rank k.

Argument function

The branch of rank k of the argument function is

$$\begin{array}{ccc} \mathbb{C}\backslash Ox^{+} & \longrightarrow &]2k\pi, 2(k+1)\pi[\\ z & \longmapsto & \theta = \arg_k z \end{array}$$



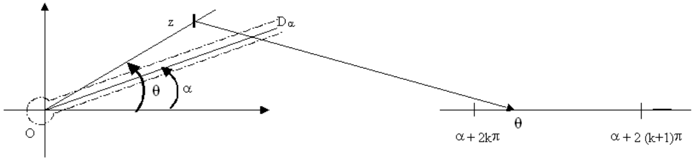
Remarks

- The half-axe Ox^{+} is called the branch cut.
- When $k = 0$, the restriction is called "principal branch".

Argument function: other definition (more general)

The branch of rank k of the argument function is

$$\begin{array}{rcl} \mathbb{C}\backslash D_\alpha & \longrightarrow &]\alpha + 2k\pi, \alpha + 2(k+1)\pi[\\ z & \longmapsto & \theta = \arg_{k,\alpha} z \end{array}$$



Remarks

- With this definition, the half-line (or ray) D_α with origin O and angle α is the **branch cut**.

Multifunctions

Definitions

- Continuity values: values on the upper and lower sides of the branch cut.
- The point O at the origin of the branch cut is called the **branch point**.

Remarks

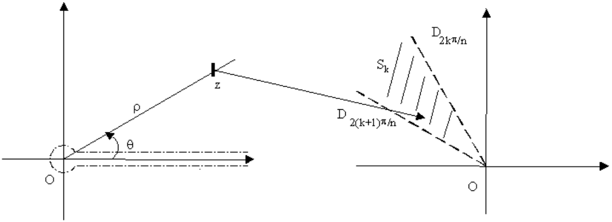
- How to represent the branch cut?
- Closed paths enclosing the branch point $\rightarrow$ branch change [WARNING]
- Closed paths enclosing the branch point $\rightarrow$ no branch change

Power functions

The branch of rank k of $z \mapsto z^{\frac{1}{n}}$ is

$$\left\{ \begin{array}{rcl} \mathbb{C} \setminus Ox^+ & \to & S_k \\ z & \mapsto & z_{(k)}^{\frac{1}{n}} = |z|^{\frac{1}{n}} \; e^{i\frac{1}{n}\arg_k(z)} = \rho^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{i\frac{2k\pi}{n}} \end{array} \right. \qquad \theta \in]0, 2\pi[$$

This mapping is a bijective function from $\mathbb{C} \setminus Ox^+$ in the open circular sector S_k delimited by the two lines $D_{\frac{2k\pi}{n}}$ and $D_{\frac{2(k+1)\pi}{n}}$ coming from O and making angles $\frac{2k\pi}{n}$ and $\frac{2(k+1)\pi}{n}$ with Ox^+.



Power functions

Extensions

- Function $z \mapsto (z-a)^{\frac{1}{n}}$.
- Function $z \mapsto (z-a)^{\alpha}$, $\alpha \in \mathbb{R}$.

Example

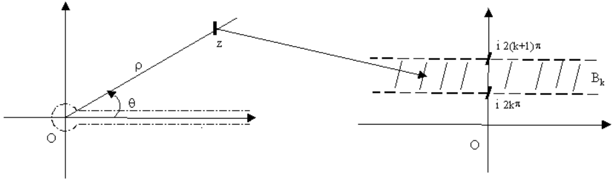
Restriction of $z \mapsto (z+1)^{\frac{1}{2}}$.

Logarithm function

The restriction of rank **k** of $z \mapsto \log(z)$ is

$$\left\{ \begin{array}{rcll} \mathbb{C} \setminus Ox^+ & \to & B_k & \\ z = |z|e^{i\theta + i2k\pi} & \mapsto & \log_k(z) & = \ln|z| + \arg_k(z) \\ & & & = \ln\rho + i\theta + i2k\pi \end{array} \right.$$

where B_k is the open strip-like set defined by:
$\{z \;/\; \mathrm{Im} z \in]2k\pi, 2(k+1)\pi[\}$.



Extension

- Function $z \mapsto z^\alpha$, $\alpha \in \mathbb{C}$ defined by $z_k^\alpha = e^{\alpha \log_k(z)}$.

Outline

Differentiable functions of two variables

A function $P(x, y)$ is differentiable at the point (x_0, y_0) when it is defined in an open set containing this point and:

$$\Delta P = A(x_0, y_0)\, h + B(x_0, y_0)\, k + \|(h, k)\|\, \varepsilon(h, k)$$

with

$$\Delta P = P(x_0 + h, y_0 + k) - P(x_0, y_0)$$

et

$$\lim_{\|(h,k)\| \to 0} \varepsilon(h, k) = 0$$

Outline

Definition of the differentiability

Definition

$f(z)$ differentiable at z_0 if and only if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. It is denoted

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example 1

$f(z) = z$

$$\lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1, \text{ hence } f \text{ est dérivable en } z_0.$$

Example 2

$f(z) = z^2$

$$\lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0, \text{ hence } f \text{ est dérivable en } z_0.$$

Definition of the differentiability

Counter example

$g(z) = \overline{z}$

$$\begin{aligned} \frac{\overline{z} - \overline{z_0}}{z - z_0} &= \frac{(x - x_0) - i\,(y - y_0)}{(x - x_0) + i\,(y - y_0)} \\ &= \frac{1 - i\,\frac{y - y_0}{x - x_0}}{1 + i\,\frac{y - y_0}{x - x_0}} = \frac{1 - im}{1 + im} \end{aligned}$$

which depends on the slope m of the path, thus

$$\lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \text{ does not exist}$$

$\Rightarrow f$ is not differentiable at z_0.

Necessary and sufficient condition

Property

A complex variable function f is differentiable at the point $z_0 = x_0 + iy_0$ if and only if

- $P(x, y)$ et $Q(x, y)$ are differentiable at the point (x_0, y_0) and
- the Cauchy conditions are fulfilled:

$$\begin{cases} \frac{\partial P}{\partial x}(x_0, y_0) & = & \frac{\partial Q}{\partial y}(x_0, y_0) \\ \frac{\partial P}{\partial y}(x_0, y_0) & = & -\frac{\partial Q}{\partial x}(x_0, y_0) \end{cases}$$

Remark

The demonstration of this condition allows ones to obtain

$$\begin{aligned} f'(z_0) &= \frac{\partial P}{\partial x}(x_0, y_0) + i\frac{\partial Q}{\partial x}(x_0, y_0) \\ f'(z_0) &= \frac{\partial Q}{\partial y}(x_0, y_0) - i\frac{\partial P}{\partial y}(x_0, y_0) \end{aligned}$$

Outline

Holomorphic functions

Definition

A complex variable function is said holomorphic on an open set A of $\mathbb{C}$ if it is differentiable in any point of A. Notation: $f \in \mathcal{H}/A$.

Properties

The properties are the same as those related to differentiable in $\mathbb{R}$. Let define f and $g \in \mathcal{H}/A$.

- $\lambda f + \mu g \in \mathcal{H}/A$ et $(\lambda f + \mu g)' = \lambda f\,' + \mu g'$
- $fg \in \mathcal{H}/A$ et $(fg)' = f\,'g + fg'$
- If $\forall z \in A, g(z) \neq 0$, then:

$$\frac{1}{g} \in \mathcal{H}/A \text{ et } \left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

- If $f \in \mathcal{H}/A$, $g \in \mathcal{H}/f(A)$, then: $(g \circ f) \in \mathcal{H}/A$ et $(g \circ f)' = (g' \circ f)\, f\,'$
- If f is bijective from A onto $f(A)$, then:

$$f^{-1} \in \mathcal{H}/f(A) \text{ et } \left(f^{-1}\right)' = \frac{1}{f\,' \circ f^{-1}}$$

Differentiability of usual functions

Algebraic functions

One formally differentiates with respect to z as for the real variable function with respect to x :

$$\begin{aligned} (az)' &= a \\ (z^m)' &= mz^{m-1}, \quad m \in \mathbb{Z} \end{aligned}$$

Functions defined by series expansion

Theorem of differentiability of power series:

The function $f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots$ of convergence radius R is holomorphic on the open disk $d(O, R)$. Its derivative is the sum of the term-wise differential series. Thus

$$\begin{aligned} (e^z)' &= e^z \\ (chz)' &= shz \\ (\cos z)' &= -\sin z \\ & \text{etc} \ldots \end{aligned}$$

One derives with respect to z as one derives in R with respect to x.

Differentiability of multifunctions

- **Derivative of** $\log_k z$

$$Z = \log_k(z) = \ln\rho + i\theta + 2ik\pi$$

defined from $\mathbb{C} \setminus Ox^+$ to B_k.
One reminds that $\exp(\log_k(z)) = z$. By the reciprocal formula, the derivative is given

$$\begin{aligned} z &= f(Z) \Longrightarrow z' = f'(Z) \\ Z &= f^{-1}(z) \Longrightarrow Z' = \frac{1}{f'(f^{-1}(z))} \end{aligned}$$

Thus:

$$\begin{aligned} z &= \exp(Z) \Longrightarrow z' = \exp(Z) \\ Z &= \log_k(z) \Longrightarrow Z' = \frac{1}{\exp(\log_k(z))} = \frac{1}{z} \end{aligned}$$

The additive constant disappears. Thus:

$$\boxed{\log_k z \text{ holomorphic on } \mathbb{C} \setminus Ox^+ \text{ et } (\log_k)'(z) = \frac{1}{z}}$$

Differentiability of multifunctions

- **Derivative of $z_{(k)}^{\alpha}, \alpha \in \mathbb{C}$**

$$z_{(k)}^{\alpha} = \exp\left(\alpha \log_k(z)\right)$$

By differentiability of compound functions, one obtains:

$$\left[z_{(k)}^{\alpha}\right]' = \left[\alpha \left[\log_k(z)\right]\right]' \exp\left[\alpha \log_k(z)\right]$$

Thus:

$$\left[z_{(k)}^{\alpha}\right]' = \frac{\alpha}{z} z_{(k)}^{\alpha}$$

The derivative owns the same multiplicative constant. Thus

$$\boxed{z_{(k)}^{\alpha} \text{ holomorphic on } \mathbb{C} \setminus Ox^{+} \text{ et } \left[z_{(k)}^{\alpha}\right]' = \frac{\alpha}{z} z_{(k)}^{\alpha}}$$

Outline

Complement : harmonic functions

If we had time...

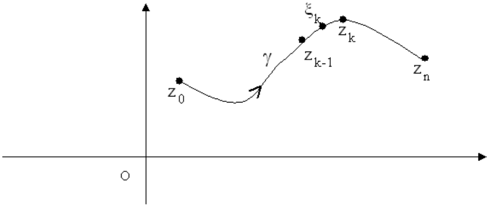
Outline

Path

- A **path** of $\mathbb{C}$ is continuous function $\gamma \ : [a, b] \rightarrow \mathbb{C}$, where $[a, b]$ is an interval of $\mathbb{R}$.
- If $\gamma(a) = \gamma(b)$, γ is a **closed path**.
- γ is piecewise C^1 if $\gamma'(t)$ exists and is continuous on the intervals $[t_{j-1}, t_j]$ of $\mathbb{R}$ with $t_0 = a < t_1 < ... < t_n = b$.

Complex curvilinear integral

Let $f(z)$ a function defined on a path γ which is piecewise-C^1



Let $\bigcup_{k=1}^{n} \widehat{z_{k-1}z_k}$ define a subdivision of this path with $\xi_k \in \widehat{z_{k-1}z_k}$, $z_k = \gamma(t_k)$, $z_0 = \gamma(a)$ and $z_n = \gamma(b)$.

Definition:

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$
$$\text{with } \max_k |z_k - z_{k-1}| \underset{n\to\infty}{\rightarrow} 0$$

Complex curvilinear integral

With the following notations

$$\begin{aligned} z_k &= x_k + iy_k \\ z_k - z_{k-1} &= \Delta x_k + i\Delta y_k \\ \xi_k &= a_k + ib_k \\ f(\xi_k) &= P(a_k, b_k) + iQ(a_k, b_k) \end{aligned}$$

it yields

$$\begin{aligned} \int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} P(a_k, b_k)\Delta x_k - Q(a_k, b_k)\Delta y_k \\ +i \lim_{n\to+\infty} \sum_{k=1}^{n} Q(a_k, b_k)\Delta x_k + P(a_k, b_k)\Delta y_k \end{aligned}$$

with $\max_k |\Delta x_k| \to 0$ and $\max_k |\Delta y_k| \to 0$. Hence

$$\int_\gamma f(z)dz = \int_\gamma (Pdx - Qdy) + i \int_\gamma (Qdx + Pdy)$$

Complex curvilinear integral

Sufficient condition of existence

P and Q continious on γ
or f continious on γ

In practice: γ is parametrized

$$\int_{\gamma} f(z)dz = \int_{a}^{b} f\left(\gamma\left(t\right)\right)\gamma'\left(t\right)dt$$

Usual paths

- Line segment parallel to the X-axis,
  $z = x + iy_0,\ x \in [x_1, x_2]$
- Line segment parallel to the Y-axis,
  $z = x_0 + iy,\ y \in [y_1, y_2]$
- Arc of radius R_0
  $z = R_0 e^{i\theta},\ \theta \in [\theta_1, \theta_2]$
- Line segment coming from the origin
  $z = \rho e^{i\theta_0},\ \rho \in [\rho_1, \rho_2]$

Complex curvilinear integral

Elementary properties of integrals

a) Linearity

$$\int_\gamma (\lambda f(z) + \mu g(z))dz = \lambda \int_\gamma f(z)dz + \mu \int_\gamma g(z)dz$$

b) Sense of the pathγ

$$\int_{\gamma^-} f(z)dz = -\int_{\gamma^+} f(z)dz$$

$\gamma^- = \gamma^+$ followed in the reverse sense.

c) Integral of a constant $f(z) = K$

$$\sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) = (z_n - z_0)K = (\gamma(b) - \gamma(a))K$$

Outline

Jordan lemmas

1st Lemma Jordan

Assumptions

$C_r(a, r)$ arc of center a and radius r

$\lim_{r \to 0(\text{ resp. } \infty)} \sup_{C_r} \ |(z-a)\, f(z)| = 0$

Conclusion

$$\boxed{\lim_{r \to 0(\text{ resp. } \infty)} \int_{C_r} f(z) dz = 0}$$

Proof:

$$\begin{aligned} \left| \int_{C_r} f(z) dz \right| &= \left| \int_{\alpha}^{\beta} f(a + re^{i\theta}) rie^{i\theta} d\theta \right| \\ &\leqslant \int_{\alpha}^{\beta} \left| rf(a + re^{i\theta}) \right| d\theta \\ &\leqslant (\beta - \alpha) \sup_{C_r} \ |(z-a)\, f(z)| \end{aligned}$$

Jordan lemmas

2nd Jordan lemmas

Assumption

$\lim_\infty \sup_{C_r} |f(z)| = 0$

Conclusions

$$\begin{array}{|ll|}\hline \lim_\infty \int_{C_r} e^{imz} f(z)dz = 0 & \text{pour } m > 0 \text{ et } C_r = C_r^+ \\ \lim_\infty \int_{C_r} e^{imz} f(z)dz = 0 & \text{pour } m < 0 \text{ et } C_r = C_r^- \\ \lim_\infty \int_{C_r} e^{mz} f(z)dz = 0 & \text{pour } m < 0 \text{ et } C_r = C_r^d \\ \lim_\infty \int_{C_r} e^{mz} f(z)dz = 0 & \text{pour } m > 0 \text{ et } C_r = C_r^g \\ \hline \end{array}$$

Proof:

$$\begin{aligned} |I_r| &= \left| \int_{C_r} e^{imz} f(z)dz \right| = \left| \int_0^\pi e^{imre^{i\theta}} f(re^{i\theta}) ire^{i\theta} d\theta \right| \\ &\leq 2r \sup_{C_r} |f(z)| \int_0^{\frac{\pi}{2}} e^{-mr\sin\theta} d\theta \\ &\leq \sup_{C_r} |f(z)| \frac{\pi}{m}(1 - e^{-mr}) \ (\text{car } \sin\theta \geqslant \frac{2\theta}{\pi}) \end{aligned}$$

Outline

Cauchy theorem

1-connected (or simply connected) domain

Assumptions

f holomorphic on Ω, non-null open space of $\mathbb{C}$

Let $D \subset \Omega$ define a simply-connected domain of contour C

Conclusion

$$\boxed{\int_C f(z)dz = 0}$$

Proof (by the use of the Green-Riemann formula)

$$\int_{C^+} Adx + Bdy = \int\int_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy$$

Cauchy theorem

n-connected domain - Generalization

Example of a 2-connected domain

$$\int_C f(z)dz = \int_{C_1^+} f(z)dz + \int_{C_2^-} f(z)dz = 0$$

Oriented contour
$\overrightarrow{\tau}$ tangent vector
$\overrightarrow{n}$ oriented interior normal

$$(\overrightarrow{\tau}, \overrightarrow{n}) = +\frac{\pi}{2}$$

For $\delta D = C_1^+ \cup C_2^-$, it yields

$$\int_{\delta D} f(z)dz = 0$$

Cauchy theorem
Application

Let f define a holomorphic function on a 1-connected domain D.

a) Definition of $\int_a^b f(z)dz$

Let a and b define two points of D.

Let γ_1, γ_2 define two paths inside D with origin a and end point b. Then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz = \int_a^b f(z)dz$$

b) Definition of $F_{z_0}(u) = \int_{z_0}^u f(z)dz$, $u \in \mathbb{C}$

$F_{z_0}(u)$ is independent of the path from z_0 to u included in D

$F_{z_0}(u)$ is a primitive of $f(z)$ such that $F'_{z_0}(u) = f(u)$.

Outline

Residue theorem

Assumptions

- f holomorphic on $\Omega \backslash \bigcup_j z_j$, Ω non-empty open set of $\mathbb{C}$
- z_j isolated singularities of f
- $D \subset \Omega$ 1-connected domain of contour ∂D inside Ω

Conclusion

$$\boxed{\int_{\partial D^+} f(z)dz = 2i\pi \sum_{z_j \in D} \text{res}f(z_j)}$$

with (definition of $\text{res}f(z_j)$) :

$$\boxed{\text{res}f(z_j) = \lim_{r \to 0} \frac{1}{2i\pi} \int_{C^+(z_j,r)} f(z)dz}$$

Remarks and definition

- **Isolated singularities (IS, or isolated singular point)**
  z_j is an IS of $f(z)$ if and only if $\exists r > 0$ such that f is holomorphic in $d(z_j, r) \backslash \{z_j\}$, where $d(z_j, r)$ stands for the disc of center z_i and radius r_i.
- **Computing the residue thanks to the Laurent series**
  If z_j is an IS, one admits that f has a Laurent series in $d(z_j, r) \backslash \{z_j\}$:

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_j)^n} + \sum_{n=0}^{\infty} a_n (z - z_j)^n$$

Thus, it comes:

$$\int_{C^+(z_j,r)} f(z)dz = \sum_{n=1}^{\infty} \int_{C^+} \frac{b_n}{(z - z_j)^n} dz + \sum_{n=0}^{\infty} \int_{C^+} a_n (z - z_j)^n dz$$

Remarks and definition

We set $z - z_j = re^{i\theta}$ and it yields

$$\sum_{n=1}^{\infty} \int_0^{2\pi} \frac{b_n i d\theta}{r^{n-1} e^{i(n-1)\theta}} + i \sum_{n=0}^{\infty} \int_0^{2\pi} a_n r^{n+1} e^{i(n+1)\theta} d\theta$$

All the integrals are null (straightforward...) except:

$$\int_0^{2\pi} \frac{b_n i d\theta}{r^{n-1} e^{i(n-1)\theta}} \quad \text{with} \quad n = 1$$

Thus :

$$\int_{C^+(z_j, r)} f(z) dz = \int_0^{2\pi} b_1 i d\theta = 2i\pi b_1$$

Conclusion : res$f(z_j)$ is the coefficient of the term $\frac{1}{z - z_j}$ of the main part of the Laurent series of f.

Remarks and definition

- **Computing the residue in case of a pole of order p**
  One computes the Taylor series of $\varphi(z) = (z - z_j)^p f(z)$ which is holomorphic in $V(z_j)$

$$\varphi(z) = \varphi(z_j) + \ldots + \frac{(z - z_j)^{p-1}}{(p-1)!}\varphi^{(p-1)}_{(z_j)} + \ldots$$

  As a consequence, the Laurent series of f is:

$$f(z) = \frac{\varphi(z_j)}{(z - z_j)^p} + \ldots + \frac{\varphi^{(p-1)}_{(z_j)}}{(p-1)!(z - z_j)} + \ldots$$

  thus

$$\boxed{\mathrm{res} f(z_j) = \tfrac{1}{(p-1)!}\varphi^{(p-1)}_{(z_j)} = \tfrac{1}{(p-1)!}\ \tfrac{d^{p-1}}{dz^{p-1}}\left[(z - z_j)^p f(z)\right]\Big|_{z=z_j}}$$

In practice:

- for $p > 2$, one compute the Laurent series,
- for $p = 2$, one can use $\mathrm{res} f(z_j) = \frac{d}{dz}(z - z_j)^2 f(z)\Big|_{z=z_j}$,
- for $p = 1$, one has $\mathrm{res} f(z_j) = \lim\limits_{z \to z_j}(z - z_j)f(z)$

Remarks and definition

Interesting particular case : z_j pole of order 1, $f(z) = \frac{P(z)}{Q(z)}$, $P(z_j) \neq 0$
One expands $Q(z)$:

$$Q(z) = 0 + (z - z_j)Q'(z_j) + \frac{(z - z_j)^2}{2!}Q''(z_j) + ...$$

thus

$$\boxed{\lim_{z \to z_j}(z - z_j)f(z) = \frac{P(z_j)}{Q'(z_j)}}$$

This formula is interesting for some residue calculus, such as $f(z) = \frac{1}{\sin z}$ en $z = 0$. Indeed:

$$\text{res}f\,(0) = \frac{P(0)}{Q'(0)} = \frac{1}{\cos 0} = 1$$

Outline

Integrals of the form: $I = \int_{-\infty}^{\infty} f(x)dx$

Very often, one defines $f(z)$ and the contour which consists of a straight line associated with I and a circular parts which close path. *Example:*

Computing

$$I = \int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1} dx$$

Integrals defined by a multifunction

Example: show that, for $a \in]0, 1[$

$$J = \int_0^{\infty} \frac{x^{a-1}}{1+x} dx = \frac{\pi}{\sin(\pi a)}$$

Trigonometric integrals

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta$$

where R is a rational fraction. One sets $z = e^{i\theta}$ and one derives $\cos\theta$ and $\sin\theta$ as functions of z.
It consists of computing an integral on the unit circle.
Example **:** show that

$$J = \int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = \frac{\pi}{2}$$

Outline

Application to the sum of a series

See exercise session.

Outline

Definition

Set of the (Laplace) transformable functions

E is the set of the functions f defined on $\mathbb{R}^+$ such that

- f is locally integrable, i.e., $\int_0^A f(t)dt < \infty, \forall A$
- It exists x_0 such that $\int_0^\infty e^{-x_0 t} f(t)dt < \infty$

Laplace transform

For $f \in E$, one defines its Laplace transform as

$$\boxed{F(p) \triangleq \int_0^\infty e^{-pt} f(t)dt \qquad p \in \mathbb{C}}$$

Notation: $F(p) = TL(f(t))$

Definition
Convergences

(simple) Convergence

Thereom 1
If $F(p)$ exists for $p = p_0 = x_0 + iy_0$ then
$F(p)$ exists $\forall p$ such that $\mathrm{Re}p > \mathrm{Re}p_0 = x_0$

Consequence : $\{x \in \mathbb{R}, F(p) < \infty\}$ admits a lower bound denoted x_c and called abscissa of (simple) convergence of F.

Absolute convergence

Theorem 2
If $\int_0^\infty \left|e^{-pt}f(t)\right| \, dt$ exists for $p = p_0 = x_0 + iy_0$ then
$\int_0^\infty \left|e^{-pt}f(t)\right| \, dt$ exists $\forall p$ such that $\mathrm{Re}p > \mathrm{Re}p_0 = x_0$

Consequence : $\left\{x \in \mathbb{R}, \int_0^\infty \left|e^{-pt}f(t)\right| \, dt < \infty\right\}$ admits a lower bound denoted x_{ca} and called abscissa of absolute convergence of F (obviously, $x_c \leq x_{ca}$)

Example: $f(t) = e^{kt} \sin\left[e^{kt}\right], k > 0$, $x_c = 0$ and $x_{ca} = k$.
Remark: one often has $x_c = x_{ca}$.

Definition

Fundamental theorem

If $f(t)$ is piecewise continuous on $\mathbb{R}^+$,
then $F(p) = \int_0^\infty e^{-pt} f(t) dt$ is holomorphic on $]x_c, +\infty[$ and then it is infinitely differentiable on $]x_c, +\infty[$ with

$$\frac{d^n F(p)}{dp^n} = \int_0^\infty \frac{d^n}{dp^n} \left[e^{-pt} f(t)\right] dt$$

Consequence: deriving x_c from $F(p)$

If $F(p)$ a function of the complex variable p is the Laplace transform of a function $f(t)$ which admits isolated singularities s_k and branching points r_j in $\mathbb{C}$, then $x_c = \sup \mathrm{Re}(s_k, r_j)$

Examples: $F(p) = \frac{1}{p(p-2)}$ $\quad x_c = 2$

$F(p) = \frac{1}{p+1}$ $\quad x_c = 0$

Outline

Usual properties

a) Linearity

$$TL\,(\lambda f + \mu g) = \lambda F(p) + \mu G(p)$$

Generally, abscissa of convergence $x_c = \sup(x_{c_f}, x_{c_g})$.

b) Derivation

*** with respect to** p

$$TL\,\{(-1)^n t^n f(t)\} = \frac{d^n}{dp^n} F(p)$$

*** with respect to t** (f continuous on $[0, +\infty[$)

$$TL\,[f'(t)] = pF(p) - f(0^+)$$

Generalization:

$$TL\left[f^{(n)}(t)\right] = p^n F(p) - p^{n-1} f(0^+) - ... - f^{(n-1)}(0^+)$$

Application: resolution of linear differential equations

Usual properties

c) Integration
*** LT of a primitive**

$$TL\left[\int_0^t f(u)du\right] = \frac{F(p)}{p}$$

Abscissa of convergence: $\sup(x_c, 0)$
*** Primitive of a LT**

$$TL\left[\frac{f(t)}{t}\right] = \int_p^{\infty} F(u)du$$

Usual properties

d) Translation
*** with respect to p**

$$TL\left[e^{at}f(t)\right] = F(p-a)$$

Abscissa of convergence: $x_c + \mathrm{Re}(a)$
*** with respect to t**

$$TL\left[f(t-a)U(t-a)\right] = e^{-ap}F(p)$$

Abscissa of convergence: x_c
Remark: Application to periodic functions
e) Scaling

$$TL\left[f\left(\frac{t}{k}\right)\right] = kF(kp) \qquad k > 0$$

Abscissa of convergence: $\frac{x_c}{k}$

Usual properties

f) Convolution

$$TL\left[\int_0^t f(u)g(t-u)du\right] = F(p)G(p)$$

g) Theorems of the initial and final values

$$\lim_{t\to 0^+} f(t) = \lim_{p\to\infty} pF(p)$$

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p)$$

h) Transform of series

Series of general term $a_n \frac{t^n}{n!}$
with abscissa of convergence $R_c = \infty$

$$TL\left[\sum_{n=1}^{\infty} a_n \frac{t^n}{n!}\right] = \sum_{n=1}^{\infty} \frac{a_n}{p^{n+1}}$$

Example: show that $TL\left[\frac{\sin\omega t}{t}\right] = Arctg\,\frac{\omega}{p}$

Use two methods: series expansion and $TL\left[\frac{x(t)}{t}\right]$

Some Laplace transforms

Function	TL	Convergence
$U(t)$	$\frac{1}{p}$	$x_c = 0$
$e^{\alpha t}$	$\frac{1}{p-\alpha}$	$x_c = \mathrm{Re}\alpha$
$e^{i\omega t}$	$\frac{1}{p-i\omega}$	$x_c = 0$
$ch\,(\alpha t)$	$\frac{p}{p^2-\alpha^2}$	$x_c = \sup \mathrm{Re}(\alpha, -\alpha)$
$sh\,(\alpha t)$	$\frac{\alpha}{p^2-\alpha^2}$	$x_c = \sup \mathrm{Re}(\alpha, -\alpha)$
$\cos \omega t$	$\frac{p}{p^2+\omega^2}$	$x_c = 0$
$\sin \omega t$	$\frac{\omega}{p^2+\omega^2}$	$x_c = 0$
t	$\frac{1}{p}$	$x_c = 0$
$t^n, n \in \mathbb{N}$	$\frac{n!}{p^{n+1}}$	$x_c = 0$
$t^\alpha, \alpha \in \mathbb{R}$	$\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$	

with $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ et $\Gamma(n+1) = n\Gamma(n) = n!$

Outline

Inversion formula

$$X(p) = \int_0^{\infty} x(t)e^{-pt}dt = \int_0^{\infty} x(t)e^{-at}e^{-j2\pi ft}dt$$

with $p = a + j2\pi f$

Analogy with the Fourier transform

$$X(f) = TF(x(t)) = \int_{\mathbb{R}} x(t)e^{-j2\pi ft}$$
$$x(t) = TF^{-1}(X(f)) = \int_{\mathbb{R}} X(f)e^{+j2\pi ft}df$$

Hence :

$$X(p) = TF\left[x(t)e^{-at}U(t)\right]$$

and thus the inversion formula:

$$\boxed{x(t)U(t) = \frac{1}{2i\pi}\int_{D\uparrow} X(p)e^{pt}dp}$$

One applies the residue theorem with $X(p)e^{pt}$.
Example: $X(p) = \frac{1}{\sqrt{p}}$

Outline

Differential equations with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_n y(t) = f(t)$$

Initial conditions

$$y(0) = b_0, y'(0) = b_1, ..., y^{(n-1)}(0) = b^{(n-1)}$$

Laplace transform

$$\begin{gathered} TL\left[a_n y(t)\right] = a_n Y(p) \\ TL\left[y^{(n)}(t)\right] = p^n Y(p) - p^{n-1} y(0^+) - ... - y^{(n-1)}(0^+) \\ TL\left[\Omega_n(y)\right] = \Omega_n(p) Y(p) - Q_{n-1}(p) \\ TL\left[f(t)\right] = F(p) \end{gathered}$$

Algebraic problem

$$\Omega_n(p) Y(p) = Q_{n-1}(p) + F(p)$$

$$Y(p) = \frac{Q_{n-1}(p)}{\Omega_n(p)} + \frac{F(p)}{\Omega_n(p)} = Y_1(p) + Y_2(p)$$

Differential equations with constant coefficients

a) $Y_1(p)$ **Algebraic fraction**

$$Y_1(p) = \frac{Q_{n-1}(p)}{\prod_{i=1}^{r}(p-p_i)^{k_i}}$$

where p_i is a k_i-order root with $\sum_{i=1}^{r} k_i = n$

Partial fraction decomposition :

$$Y_1(p) = \sum_{i=1}^{r}\left\{\frac{A_{i1}}{p-p_i} + \frac{A_{i2}}{(p-p_i)^2} + \ldots + \frac{A_{ik_i}}{(p-p_i)^{k_i}}\right\}$$

where

$$y_1(t) = \sum_{i=1}^{r} e^{p_i t}\left[A_{i1} + A_{i2}t + \ldots + A_{ik_i}t^{k_i-1}\right]$$

b) $Y_2(p) = \frac{F(p)}{\Omega_n(p)} = F(p) \times \frac{1}{\Omega_n(p)}$ thus:

$$y_2(t) = \int_0^t f(u)R_n(t-u)du$$

Hence, the solution of the problem is $y(t) = y_1(t) + y_2(t)$

Partial differential equation of several variables

The LT allows one to reduce the equation with respect to one dimension.
Example:
Two-dimensional spatio-temporal problem: string vibration $f(x,t)$

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

Initial conditions

$$f(x,0) = \varphi(x)$$
$$\frac{\partial f}{\partial t}(x,0) = \psi(x)$$

Conditions at limits

$$f(\infty,t) = 0$$
$$f(0,t) = g(t)$$

Partial differential equation of several variables

Solution thanks to the LT (p is a considered as a parameter)

$$F(x,p) = \int_0^{\infty} e^{-pt} f(x,t) dt$$

$$\begin{aligned} TL\left[\frac{\partial f}{\partial t}\right] &= pF(x,p) - f(x,0) \\ &= pF(x,p) - \varphi(x) \end{aligned}$$

$$\begin{aligned} TL\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right] &= p^2 F(x,p) - pf(x,0) - \frac{\partial f}{\partial t}(x,0) \\ &= p^2 F(x,p) - p\varphi(x) - \psi(x) \end{aligned}$$

$$\begin{aligned} TL\left[\frac{\partial^2 f}{\partial x^2}(x,t)\right] &= \int_0^{\infty} e^{-pt} \frac{\partial^2 f(x,t)}{\partial x^2} dt \\ &= \frac{\partial^2}{\partial x^2} \int_0^{\infty} e^{-pt} f(x,t) dt = \frac{d^2 F(x,p)}{dx^2} \end{aligned}$$

Partial differential equation of several variables

One obtains

$$\frac{d^2F(x,p)}{dx^2} - p^2 F(x,p) = p\varphi(x) + \psi(x)$$

with

$$\begin{aligned} F(\infty, p) &= TL\left[f(\infty, t)\right] = 0 \\ G(p) &= TL\left[g(t)\right] = TL[f(0,t)] = F(0,p) \end{aligned}$$

One-dimensional problem (differential equations + conditions at limits).

Outline

Definition

Definition

One defines the Z transform of a series $x(n), n \in \mathbb{Z}$ as:

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} \qquad z \in \mathbb{C}$$

Notation:

$$X(z) = TZ(x(n))$$

Remark: bilateral and unilateral TZ.

Definition

Domain of convergence

The domain of convergence is the set of complex numbers z such that the series $X(z)$ converges.

Reminder: Cauchy criterion

$$\lim_{n\to+\infty} \sqrt[n]{|u_n|} < 1 \Longrightarrow \sum_{n=0}^{+\infty} u_n \text{ converge}$$

One has a sufficient condition of convergence. Thanks to this criterion, one shows that the series $X(z)$ converges once:

$$0 \leq R_x^- < |z| < R_x^+ \leq +\infty$$

Example: $X(z) = \sum_{n=0}^{+\infty} z^{-n}$ converges for $|z| > 1$

Outline

Usual properties

Linearity

$$TZ(ax(n) + by(n)) = aX(z) + bY(z)$$

Convergence: if $R^+ = \min(R_x^+, R_y^+)$ and $R^- = \max(R_x^-, R_y^-)$, then the convergence domain contains $]R^-, R^+[$.

Shifting

$$TZ(x(n - n_0)) = z^{-n_0} X(z)$$

Same domain of convergence as $X(z)$.

Scaling

$$TZ(a^n x(n)) = X\left(\frac{z}{a}\right)$$

Domain of convergence: $|a|\, R_x^- < |z| < |a|\, R_x^+$

Usual properties

Differentiability

The Z transform defines a Laurent series which is infinitely differentiable term-by-term in its domain of convergence. Thus

$$TZ(nx(n)) = -z\frac{dX(z)}{dz}$$

Same domain of convergence as $X(z)$.

Convolution product

The convolution between the series $x(n)$ and $y(n)$ is defined as:

$$u(n) = x(n) * y(n) = \sum_{k=-\infty}^{+\infty} x(k)y(n-k)$$

Thus

$$TZ(x(n) * y(n)) = X(z)Y(z)$$

The domain of convergence of $U(z)$ can be larger than the intersection of domains of convergence of $X(z)$ and $Y(z)$, respectively.

Outline

Inverse Z transform

The inverse Z transform is given by:

$$x(n) = \frac{1}{j2\pi} \int_{C^+} X(z) z^{n-1} dz$$

where C is a closed path included into the domain of convergence

TZ inverse

Proof

One has to compute the integrals

$$J(n,k) = \int_{C^+} z^{n-k-1} dz$$

Thanks to the residue theorem, one shows that:

$$J(n,k) = \begin{cases} 0 \text{ si } n \neq k \\ j2\pi \text{ si } n = k \end{cases}$$

Hence:

$$\begin{aligned} \frac{1}{j2\pi} \int_{C^+} X(z) z^{n-1} dz &= \frac{1}{j2\pi} \int_{C^+} \left(\sum_{k=-\infty}^{\infty} x(k) z^{-k} \right) z^{n-1} dz \\ &= \frac{1}{j2\pi} \sum_{k=-\infty}^{\infty} x(k) J(n,k) \\ &= x(n) \end{aligned}$$

Remark : tables

Outline

Discrete signal filtering

See exercise session and/or later.

Recurrence relations

Example: 1-st order system

$$y(n) - ay(n-1) = x(n) \qquad |a| < 1$$

The input of the system is chosen as:

$$x(n) = b^n U(n) \text{ with } |b| < 1$$

where $U(n)$ is the Heaviside step function.

- Compute $y(n)$ for $n \geq 0$ given that $y(n) = 0$ for $n < 0$.
- Determine the impulse response of the system $h(n)$ such that $y(n) = x(n) * h(n)$.

Outline

Laplace and Z transforms

Let $x(t)$ define a causal signal whose Laplace transform is:

$$X(p) = \int_0^{\infty} x(t) e^{-pt} dt$$

One samples this signal with period T and one denotes $X(z)$ its Z transform:

$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}$$

Then

$$\boxed{X(z) = \sum res \frac{X(p)}{1 - e^{pT} z^{-1}}}$$

Laplace and Z transforms

The formula of inverse Laplace transform provides

$$x(t)U(t) = \frac{1}{2i\pi}\int_{D\uparrow} X(p)e^{pt}dp$$

hence

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x(nT)z^{-n} = \sum_{n=0}^{\infty}\left[\frac{1}{2i\pi}\int_{D\uparrow} X(p)e^{pnT}dp\right]z^{-n} \\ &= \frac{1}{2i\pi}\int_{D\uparrow} X(p)\sum_{n=0}^{\infty}\left(z^{-1}e^{pT}\right)^n dp \end{aligned}$$

Once $\left|z^{-1}e^{pT}\right| < 1$, on a

$$X(z) = \frac{1}{2i\pi}\int_{D\uparrow} X(p)\frac{1}{1-z^{-1}e^{pT}}dp = \sum res \frac{X(p)}{1-e^{pT}z^{-1}}$$

Complex Variables

Laplace Transform – Z Transform

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